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A NOTE ON A NONLINEAR MINIMAX LOCATION PROBLEM ON A TREE NETWORK

Research Report 76-23

by

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October, 1976

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ABSTRACT

are presented

We present some new derivations, which are rather short, and direct, of properties of a nonlinear version of a minimax tree network location problem. The properties provide necessary and sufficient conditions for optimality, a means of computing the optimum objective function value, and a means of constructing the unique optimum location.

The purpose of this note is to provide new, rather short, and direct proofs for properties, established by DEARING [4], of a quite general non-linear minimax location problem. As a natural consequence of the approach we use, we obtain as well a new family of equivalent conditions for optimality to the problem.

So as to state the problem, we suppose we are given a finite undirected tree network with positive arc lengths. We denote by T an imbedding of the given tree (e.g., a planar imbedding such as a road network) having rectifiable arcs, so that it is meaningful to speak of points on the arcs as well as at vertices.

For every pair of points x and y, $x,y \in T$, we suppose the distance between x and y, $d(x,y)$, to be well defined, as in reference 2. The distance has the customary properties for every $x,y \in T$, that $d(x,y) \geq 0$; that $d(x,y) = 0$ iff $x = y$; and that $d(x,y) = d(y,x)$; also $d(x,y) \leq d(x,z) + d(z,y)$ for every $x,y,z \in T$.

The problem of interest is as follows. Suppose "existing facilities" are at distinct vertex locations $v_1 \ldots, v_m$ in the tree, and that a "new facility" is to be located at x. For each vertex v_i, $f_i[d(x,v_i)]$ is a "cost" or a "loss" incurred, strictly proportional to the distance between x and v_i, and

$$f(x) \equiv \max\{f_i[d(x,v_i)] : 1 \leq i \leq m\} \tag{1}$$

is the maximum loss. The problem of interest is to find x^* in T to minimize f defined by (1). One may wish to employ such an approach when it is more important to provide quick service than to minimize total cost.

So as to state the problem more precisely, denote the diameter of the tree by δ. We assume f_i is a strictly increasing, continuous function, with domain $[o,\delta]$, for $1 \leq i \leq m$. Also we define $f^* \equiv f(x^*)$; continuity and compactness considerations assure the existence of a minimum of f. The assumed properties for each f_i are quite weak, compared to assumptions in earlier related literature. To the best of our knowledge, Dearing was the first to solve the problem with these assumptions: all previous work assumes the functions f_i to be linear, and is discussed by DEARING and FRANCIS [2].

Subsequent to the analysis we indicate in some detail how our results differ from Dearing's.

ANALYSIS

The following definitions facilitate the analysis:

$$M \equiv \{1, 2, \ldots, m\}$$

$$MP \equiv \{i \in M : f^* \leq f_i(\delta)\}$$

$$MS \equiv \{i \in M : f_i(\delta) < f^*\}$$

$$\alpha' \equiv \max\,[f_i(0) : i \in MP]$$

$$\alpha \equiv \max\,[f_i(0) : i \in M]$$

$$\eta' \equiv \min\,[f_i(\delta) : i \in MP]$$

$$\eta \equiv \min\,[f_i(\delta) : i \in M]$$

$$f'(x) \equiv \max\,[f_i(x) : i \in MP],\ \forall\ x \in T.$$

We remark that any $f_i \ni i \in MS$ is a secondary function in the sense that it may be deleted from the definition of f without changing f, that is,

$$f(x) = f'(x)$$

for all $x \in T$. However, functions $f_i \ni i \in MP$ are primary functions in the sense that f may be changed if any such function is deleted.

The above definitions lead to

REMARK 1. (a) $MP \neq \phi$.

(b) $\alpha' \leq \alpha \leq f^* \leq \eta'$.

(c) We may assume $\alpha < \eta$ without loss of generality.

Proof. (a) The proof is simple and we omit it.

(b) The proof is trivial except for $\alpha \leq f^*$. Let $f(x^*) = f^*$. We then have

$$f_i(0) \leq f_i[d(x^*, v_i)] \leq f(x^*) = f^*, \qquad 1 \leq i \leq m,$$

giving $\alpha \leq f^*$.

(c) If $\alpha \geq \eta$ then $\exists$ functions f_i and f_j such that $f_i(0) \geq f_j(\delta)$, thus $f_j(\delta) \leq f_i(0) \leq \alpha \leq f^*$, so the function f_j may be deleted from the definition of f without changing f.

The following remark establishes properties of functions which occur repeatedly in the subsequent analysis. The proof is straightforward but tangential to the main body of the development, and we relegate the proof (of a more extensive form of the remark) to the appendix.

REMARK 2. Let $\{j, k\} \subset M$ with $j \leq k$. Define the strictly increasing, continuous function g_{jk} with nonempty domain $[a_{jk}, b_{jk}]$ by

$$g_{jk}(z) = f_j^{-1}(z) + f_k^{-1}(z),$$

where

$$a_{jk} \equiv \max\,[f_j(0),\ f_k(0)]$$

$$b_{jk} \equiv \min\,[f_j(\delta),\ f_k(\delta)].$$

Also define L_{jk} and β_{jk} by

$$L_{jk} \equiv \begin{cases} f_j^{-1}[f_k(0)] > 0 & \text{if} \quad f_j(0) < f_k(0) \\ 0 & \text{if} \quad f_j(0) = f_k(0) \\ f_k^{-1}[f_j(0)] > 0 & \text{if} \quad f_j(0) > f_k(0) \end{cases}$$

$$\beta_{jk} = a_{jk} \qquad \text{if } d(v_j,\ v_k) \leq L_{jk}$$

$$\beta_{jk} = g_{jk}^{-1}[d(v_j,\ v_k)] \text{ if } L_{jk} < d(v_j,\ v_k).$$

(a) The condition

$$d(v_j,\ v_k) \leq g_{jk}(z),\ z \in [a_{jk},\ b_{jk}]$$

is equivalent to

$\beta_{jk} \leq z, \; z \in [a_{jk}, b_{jk}]$.

(b) If $j = k = i$, $\beta_{jk} = \beta_{ii} = f_i(0)$.

(c) If $\{j, k\} \cap MS \neq \phi$, $\beta_{jk} < f^*$.

DEFINITIONS

$$\beta' = \max\,[\beta_{jk}\colon \{j, k\} \subset MP,\ j < k] \text{ if } |MP| \geq 2$$

$$\beta' = -\infty \text{ if } |MP| = 1.$$

$$\beta = \max\,[\beta_{jk} : \{j, k\} \subset M,\ j < k]$$

$$\gamma' = \max\,[\beta_{jk} : \{j, k\} \subset MP,\ j \leq k]$$

$$\gamma = \max\,[\beta_{jk} : \{j, k\} \subset M,\ j \leq k]$$

We note that $\alpha' \leq \alpha$, $\beta' \leq \beta$, $\gamma' \leq \gamma$, $\gamma' = \max(\alpha', \beta')$, and $\gamma = \max(\alpha, \beta)$, where the identities for γ' and γ are due to Remark 2(b).

Some extra definitions are convenient. Given any $y \in T$ and nonnegative number r define $N(y, r) = \{x \in T\colon\ d(x, y) \leq r\}$, and call $N(y, r)$ a <u>neighborhood</u> with <u>center</u> y and <u>radius</u> r. Given any $u, v \in T$ define $L(u, v) = \{x \in T\colon d(u, x) + d(x, v) = d(u, v)\}$; intuitively, $L(u, v)$ is the unique imbedded path joining u and v, and has "length" $d(u, v)$.

We say that a subset S of T is <u>convex</u> (or <u>connected</u>) if $L(u, v) \subset S$ for every $u, v \in S$. HORN [5] proves a "pair-wise intersection" result for trees which, slightly modified, is the foundation of the analysis to follow. The result states that the intersection of all of the members of a finite collection of (connected) subtrees of a tree is nonempty if and only if every pair-wise intersection of subtrees is nonempty. Following Horn's result, CHAN and FRANCIS [1] prove an analogous pair-wise intersection property for an imbedded tree: the intersection of all the members of a finite collection of convex (and compact) subsets of an imbedded tree T is nonempty if and only if every pair-wise intersection is nonempty.

It is intuitively appealing, and can be proven (see Lemma 1 and Property 10 of reference 3) that any neighborhood of T is a convex (or connected) set, and is also compact given rather weak assumptions about T. Hence as a special case of the pair-wise intersection property of reference 1 we have the following lemma.

LEMMA 1. Given neighborhoods $N(y_i, r_i)$ of T, $1 \leq i \leq m$, the conditions (2), (3), and (4) below are equivalent:

$$\cap\{N(y_i, r_i): 1 \leq i \leq m\} \neq \phi \tag{2}$$

$$N(y_j, r_j) \cap N(y_k, r_k) \neq \phi, \ 1 \leq j \leq k \leq m \tag{3}$$

$$d(y_j, y_k) \leq r_j + r_k, \ 1 \leq j \leq k \leq m. \tag{4}$$

We remark that the nontrivial part of the proof of the lemma is showing (3) implies (2). (2) implies (3) trivially, and it is direct to establish the equivalence of (3) and (4).

We study the function f' in order to minimize f. It is simpler to develop the theory for f' than for f, and all such theory then applies to f. Fortunately we do not need to determine the set MP used in defining f' in order to develop the theory, as we must know f* in order to construct MP, and f* is what we are trying to find.

In order to minimize f' we study the following equivalent problem:

$$\text{minimize} \quad z$$

$$\text{subject to } f_i[d(x, v_i)] \leq z, \ i \in MP \tag{5a}$$

$$z \in [\alpha', \eta']. \tag{5b}$$

We comment that (5b) is justified by Remark 1(b).

The following lemma gives conditions equivalent to (5).

LEMMA 2. Each of the conditions (6) through (16), in conjunction with the condition $z \in [\alpha', \eta']$, is equivalent to (5):

$$\exists\, x \ni f_i[d(x, v_i)] \le z, \; i \in MP \tag{6}$$

$$\exists\, x \ni d(x, v_i) \le f_i^{-1}(z), \; i \in MP \tag{7}$$

$$\exists\, x \ni x \in N(v_i, f_i^{-1}(z)), \; i \in MP \tag{8}$$

$$\exists\, x \ni x \in \cap \{N(v_i, f_i^{-1}(z)) : i \in MP\} \tag{9}$$

$$S(z) \equiv \cap \{N(v_i, f_i^{-1}(z)) : i \in MP\} \neq \phi \tag{10}$$

$$N(v_j, f_j^{-1}(z)) \cap N(v_k, f_k^{-1}(z)) \neq \phi, \; \{j, k\} \subset MP, \; j \le k \tag{11}$$

$$d(v_j, v_k) \le f_j^{-1}(z) + f_k^{-1}(z), \; \{j, k\} \subset MP, \; j \le k \tag{12}$$

$$d(v_j, v_k) \le g_{jk}(z), \; \{j, k\} \subset MP, \; j \le k \tag{13}$$

$$\varepsilon_{jk} \le z, \; \{j, k\} \subset MP, \; j \le k \tag{14}$$

$$\alpha' \le z \tag{15-a}$$

$$\beta' \le z \tag{15-b}$$

$$\gamma' \le z. \tag{16}$$

We omit a formal proof of Lemma 2, as in most cases the equivalence of adjacent conditions is clear. We use the fact that since f_i is continuous and strictly increasing it has an inverse function f_i^{-1} which also is continuous and strictly increasing. Likewise g_{jk} has an inverse function which is continuous and strictly increasing. The equivalence of (6) and (7) requires z to be in the domain of f_i^{-1}, $z \in [f_i(0), f_i(\delta)]$, which is implied by $z \in [\alpha', \eta']$. The equivalence of (13) and (14) is due to Remark 2(a). The key equivalence in (5) through (16) is the equivalence of (10), (11), and (12), which is due to Lemma 1.

Lemma 2 gives

PROPERTY 1. (a) The set of all minima of f is nonempty and consists of $S(\gamma')$, where γ' is the minimum value of f. (b) with $z = \gamma'$, each of the conditions (5) through (16) is necessary and sufficient for optimality to the minimax problem.

<u>Proof</u> (a) From Lemma 2, since (6) implies (16) we conclude γ' is a lower bound on every value of f. Using Lemma 2 and letting $z = \gamma'$ in (6) through

(16) it follows since (16) implies (6) that γ' is the minimum value of f, and that $S(\gamma')$ is the nonempty set of all minima of f. (b) This part is immediate from (a) and Lemma 2.

Since γ' depends on MP, it generally cannot be computed prior to determining f^*. Fortunately, we shall see that $\gamma' = \gamma$; γ can be computed.

PROPERTY 2 (a) If $\gamma = \alpha = f_p(0)$, then $\gamma = f^* = \gamma'$, and $p \in MP$.

(b) If $\gamma = \beta = \beta_{st}$, with $s < t$, then $\gamma = f^* = \gamma'$ and $\{s, t\} \subset MP$.

Proof (a) Property 1 gives $f^* = \gamma'$, so that $\alpha \leq f^* = \gamma' \leq \gamma$. Thus $\gamma = \alpha$ implies $\gamma = f^* = \gamma'$. Further, we know $f_i(0) < f^*$ for $i \in MS$, so $f_p(0) = \gamma = f^*$ implies $p \in MP$. (b) Property 1 gives $\gamma' = f^*$, so $\beta' \leq \gamma'$ implies $\beta' \leq f^*$. For any β_{jk} not used in computing β', $j \in MS$ or $k \in MS$, so Remark 2(c) implies $\beta_{jk} < f^*$. Thus for every β_{jk}, $\beta_{jk} \leq f^*$, so that $\beta \leq f^*$. Thus $\beta \leq f^* = \gamma' \leq \gamma = \beta$, so $\gamma = f^* = \gamma'$. Since $\beta_{st} = f^*$, Remark 2(c) implies $\{s, t\} \cap MS = \phi$, and so $\{s, t\} \subset MP$.

Given $\gamma = f^*$ we now proceed to characterize the minima of f. Since $\gamma = \max(\alpha, \beta)$, it suffices in turn to consider the cases $\gamma = \alpha$ and $\gamma = \beta$.

PROPERTY 3. If $\gamma = f_p(0)$ for some $p \in M$, then v_p is the unique minimum of f and $p \in MP$.

Proof. Property 2 gives $\gamma = \gamma'$ and $p \in MP$. Thus, be Property 1, $S(\gamma) = S(\gamma')$ is the nonempty set of minima of f. As $\gamma' = f_p(0)$, $f_p^{-1}(\gamma') = 0$, so the definition of $S(\gamma')$ gives

$$\phi \neq S(\gamma) = S(\gamma') \subset N(v_p, f_p^{-1}(\gamma')) = \{v_p\},$$

and hence v_p is the unique minimum of f.

We now consider the remaining case where $\gamma = \beta_{st}$ for some v_s and v_t and $\gamma > \alpha$. The following preliminary remark is useful.

REMARK 3. Suppose $\gamma > \alpha$ and for some distinct v_s and v_t with $s < t$, that

$$\gamma = \beta_{st} = (f_s^{-1} + f_t^{-1})^{-1} \hat{o}[d(v_s, v_t)] = g_{st}^{-1}[d(v_s, v_t)]. \quad (17)$$

The following conclusions may be drawn.

(a) We have

$$f_s^{-1}(\gamma) + f_t^{-1}(\gamma) = d(v_s, v_t) \quad (18)$$

$$\min\,[f_s^{-1}(\gamma), f_t^{-1}(\gamma)] > 0. \quad (19)$$

(b) $\exists\, x^*,\ x^* \in L(v_s, v_t) \ni$

$$d(v_s, x^*) = f_s^{-1}(\gamma). \quad (20)$$

(c) Let $x^* \in L(v_s, v_t) \ni$ (20) holds. We have $v_s \neq x^* \neq v_t$,

$$d(x^*, v_t) = f_t^{-1}(\gamma) \quad (21)$$

and

$$N(v_s, f_s^{-1}(\gamma)) \cap N(v_t, f_t^{-1}(\gamma)) = \{x^*\}. \quad (22)$$

Proof. (a) (18) follows immediately by applying $(f_s^{-1} + f_t^{-1})$ to (17). Since $\gamma > \alpha$, $\gamma > f_s(0)$ and $\gamma > f_t(0)$, so $f_s^{-1}(\gamma) > 0$ and $f_t^{-1}(\gamma) > 0$, establishing (19).

(b) From (a) we have $0 < f_s^{-1}(\gamma) < d(v_s, v_t)$, so continuity considerations and the intermediate value theorem, as discussed in reference 2, assure the existence of x^* satisfying (20).

(c) Since $x^* \in L(v_s, v_t)$, from (a) we have

$$d(v_s, x^*) + d(x^*, v_t) = d(v_s, v_t) = f_s^{-1}(\gamma) + f_t^{-1}(\gamma). \quad (23)$$

(20) and (23) now give (21). (22) then follows from (20), (21), and the fact that $x^* \in L(v_s, v_t)$. Then (19), (20), and (21) imply $v_s \neq x^* \neq v_t$.

We now employ the remark.

PROPERTY 4. Suppose $\gamma > \alpha$, and we have $\gamma = \beta_{st}$ for some distinct v_s and v_t with $s < t$. Let $x^* \in L(v_s, v_t)$ be such that $d(v_s, x^*) = f_s^{-1}(\gamma)$: x^* is the unique minimum of f, and $v_s \neq x^* \neq v_t$. Also, $\{s, t\} \subset MP$.

Proof. Property 2 gives $\gamma = \gamma'$ and $\{s, t\} \subset MP$. Thus from Property 1, the definition of $S(\gamma')$, and Remark 3, we have $\phi \neq S(\gamma) = S(\gamma') \subset N(v_s, f_s^{-1}(\gamma)) \cap N(v_t, f_t^{-1}(\gamma)) = \{x^*\}$, and hence x^* is the unique minimum of f. Remark 3 also gives $v_t \neq x^* \neq v_s$.

Parenthetically, we observe that when $d(v_j, v_k) \leq L_{jk}$ Remark 2 gives $\beta_{jk} = \max[f_j(0), f_k(0)]$, so that $\beta_{jk} \leq \alpha$. Thus with

$$\beta^* \equiv \begin{cases} -\infty \text{ if } d(v_j, v_k) \leq L_{jk} \text{ for all } 1 \leq j < k \leq m \\ \max\{\beta_{jk}: \{j, k\} \subset M, j < k, L_{jk} < d(v_j, v_k)\} \text{ otherwise} \end{cases}$$

we have $\gamma = \max(\alpha, \beta^*)$, a fact that may possibly permit γ to be computed more efficiently than by using $\gamma = \max(\alpha, \beta)$.

To summarize our analysis, all of the basic results evolve from Lemma 2, which in turn relies upon the pair-wise intersection property of Lemma 1. Given $\gamma = \gamma'$, the equivalent conditions of Lemma 2 immediately imply that $\gamma = f^*$, and lead naturally to procedures (Properties 3 and 4) for computing the unique minimum.

Dearing studies properties of the minimax problem for more general norm-derived distances than the one we consider, and presents a number of properties, including a proof that $\gamma \leq f^*$. When distances are rectilinear between pairs of points in the plane, he uses a version of the pair-wise intersection property to show $\gamma = f^*$. For the tree problem, he points out that his analysis establishes f is strict quasiconvex, and that his analysis can be adapted to show that f has a unique minimum and provide the procedure (which we state in Properties 3 and 4) for computing the minimum.

The major difference between our development and Dearing's is the way in which all our basic results evolve naturally from Lemma 2. This evolution in turn entails proofs different from Dearing's. In addition, we believe Remark 2, Lemma 2, and Properties 1 and 2, to be new. Finally, we remark that our analysis can be used readily to establish that $\gamma = f^*$ when distances are a) rectilinear betweeen pairs of points in the plane or b) Tchebyshev between pairs of points in Euclidean p-space, $p \geq 1$: for these cases alternative global minima may exist. Properties 3 and 4 can be modified to provide procedures for constructing all alternative global minima.

APPENDIX

REMARK 2. Let $\{j, k\} \subset M$ with $j \leq k$. Define the function g_{jk} with domain $[a_{jk}, b_{jk}]$ by

$$g_{jk}(z) = f_j^{-1}(z) + f_k^{-1}(z),$$

where

$$a_{jk} \equiv \max\,[f_j(0), f_k(0)]$$

$$b_{jk} \equiv \min\,[f_j(\delta), f_k(\delta)].$$

The following assertions are true.

(a) $[a_{jk}, b_{jk}] \neq \phi$.

(b) g_{jk} is strictly increasing and continuous, and has range $[L_{jk}, U_{jk}]$, where

$$L_{jk} \equiv \begin{cases} f_j^{-1}[f_k(0)] > 0 & \text{if } f_j(0) < f_k(0) \\ 0 & \text{if } f_j(0) = f_k(0) \\ f_k^{-1}[f_j(0)] > 0 & \text{if } f_j(0) > f_k(0) \end{cases}$$

$$U_{jk} \equiv \begin{cases} f_k^{-1}[f_j(\delta)] + \delta < 2\delta & \text{if } f_j(\delta) < f_k(\delta) \\ 2\delta & \text{if } f_j(\delta) = f_k(\delta) \\ f_j^{-1}[f_k(\delta)] + \delta < 2\delta & \text{if } f_j(\delta) > f_k(\delta) \end{cases}$$

Also, $L_{jk} < \delta < U_{jk}$.

(c) $d(v_j, v_k)$ lies in or below the range of g_{jk}.

(d) The inverse function of g_{jk}, g_{jk}^{-1}, exists, is strictly increasing and continuous, has domain $[L_{jk}, U_{jk}]$ and range $[a_{jk}, b_{jk}]$.

(e) Define β_{jk} by

$$\beta_{jk} = a_{jk} \qquad \text{if } d(v_j, v_k) \leq L_{jk}.$$

$$\beta_{jk} = g_{jk}^{-1}[d(v_j, v_k)] \text{ if } L_{jk} < d(v_j, v_k).$$

The condition

$$d(v_j, v_k) \leq g_{jk}(z),\ z \in [a_{jk}, b_{jk}]$$

is equivalent to

$$B_{jk} \le z, \ z \in [a_{jk}, b_{jk}].$$

(f) If $j = k = i$, $\beta_{jk} = \beta_{ii} = f_i(0)$.

(g) If $\{j, k\} \cap MS \ne \phi$, $\beta_{jk} < f^*$.

Proof (a) For $i \in \{j, k\}$, f_i^{-1} has domain $[f_i(0), f_i(\delta)]$. As the domain of g_{jk} is the intersection of the domains of f_j^{-1} and f_k^{-1}, the domain of g_{jk} is thus $[a_{jk}, b_{jk}]$. $\alpha < \eta$ implies $a_{jk} < b_{jk}$, so $[a_{jk}, b_{jk}] \ne \phi$ by Remark 1 (c).

(b) It is well known that a sum of strictly increasing, continuous functions is strictly increasing and continuous, implying in turn that the range of g_{jk} is $[g_{jk}(a_{jk}), g_{jk}(b_{jk})]$.

Due to the similarity of the various cases of this part of the proof, we consider only the cases of $f_j(0) < f_k(0)$, and $f_j(\delta) < f_k(\delta)$.

When $f_j(0) < f_k(0) < f_j(\delta)$, $f_k(0)$ is in the range of f_j, so $L_{jk} = f_j^{-1}[f_k(0)]$ is well defined, and $0 < L_{jk} < \delta$. As $f_j(0) < f_k(0)$, $g_{jk}(a_{jk}) = g_{jk}[f_k(0)] = f_j^{-1}[f_k(0)] + f_k^{-1}[f_k(0)] = f_j^{-1}[f_k(0)] + 0 = L_{jk}$.

When $f_k(0) < f_j(\delta) < f_k(\delta)$, $f_j(\delta)$ is in the range of f_k, so $U_{jk} = f_k^{-1}[f_j(\delta)]$ is well defined and $0 < U_{jk} < \delta$. As $f_j(\delta) < f_k(\delta)$, $g_{jk}(b_{jk}) = g_{jk}[f_j(\delta)] = f_j^{-1}[f_j(\delta)] + f_k^{-1}[f_j(\delta)] = \delta + f_k^{-1}[f_j(\delta)] = U_{jk}$.

(c) As $0 < d(v_j, v_k) \le \delta$, and since we know $0 < L_{jk} < \delta < U_{jk} \le 2\delta$, the conclusion follows.

(d) Since g_{jk} is strictly increasing and continuous, it has an inverse function, g_{jk}^{-1}, which is also strictly increasing and continuous. The domain of g_{jk}^{-1} is the range of g_{jk}, and the range of g_{jk}^{-1} is the domain of g_{jk}.

(e) If $d(v_j, v_k) \le L_{jk}$, as L_{jk} is the minimum value of g_{jk} the equivalence of the two conditions is immediate. When $d(v_j, v_k) > L_{jk}$, by part (c) $d(v_j, v_k)$ is in the range of g_{jk}, in which case applying g_{jk}^{-1} to the first condition gives the second, while applying g_{jk} to the second condition gives the first.

(f) When $j = k = i$, we have $\beta_{jk} = a_{jk} = a_{ii} = f_i(0)$.

(g) When $\{j, k\} \cap MS \neq \phi$, the definition of MS gives $b_{jk} = \min[f_j(\delta), f_k(\delta)] < f^*$, so as $\beta_{jk} \leq b_{jk}$ we have $\beta_{jk} < f^*$.

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